

GALE DUALITY, BLOWUPS AND MODULI SPACES

Ana-Maria Castravet (Versailles)

with Carolina Araujo, Inder Kaur and Diletta Martinelli

UCLA, July 2026

Birational Geometry Seminar

BLOWUPS OF PROJECTIVE SPACES

QUESTION

What is the birational geometry of the blow-up $Bl_k \mathbb{P}^n$ of $\mathbb{P}^n$ at k points?

BLOWUPS OF PROJECTIVE SPACES

QUESTION

What is the birational geometry of the blow-up $Bl_k \mathbb{P}^n$ of $\mathbb{P}^n$ at k points?

- If k small, points in linearly general position $\rightsquigarrow$ toric varieties, geometry completely described by finite data

BLOWUPS OF PROJECTIVE SPACES

QUESTION

What is the birational geometry of the blow-up $Bl_k \mathbb{P}^n$ of $\mathbb{P}^n$ at k points?

- If k small, points in linearly general position $\rightsquigarrow$ toric varieties, geometry completely described by finite data
- As k grows $\rightsquigarrow$ geometry becomes increasingly intricate

BLOWUPS OF PROJECTIVE SPACES

QUESTION

What is the birational geometry of the blow-up $Bl_k \mathbb{P}^n$ of $\mathbb{P}^n$ at k points?

- If k small, points in linearly general position $\rightsquigarrow$ toric varieties, geometry completely described by finite data
- As k grows $\rightsquigarrow$ geometry becomes increasingly intricate
- Rich birational geometry already for $Bl_k \mathbb{P}^2$

BLOWUPS OF PROJECTIVE SPACES

QUESTION

What is the birational geometry of the blow-up $\mathrm{Bl}_k \mathbb{P}^n$ of $\mathbb{P}^n$ at k points?

- If k small, points in linearly general position $\rightsquigarrow$ toric varieties, geometry completely described by finite data
- As k grows $\rightsquigarrow$ geometry becomes increasingly intricate
- Rich birational geometry already for $\mathrm{Bl}_k \mathbb{P}^2$
- $n \geq 5$, $k = n + 2, n + 3$ points in linearly general position $\rightsquigarrow$ geometry still described by finite data: $\mathrm{Bl}_k \mathbb{P}^n$ is a **Mori dream space** (non-toric)

BLOWUPS OF PROJECTIVE SPACES

QUESTION

What is the birational geometry of the blow-up $\mathrm{Bl}_k \mathbb{P}^n$ of $\mathbb{P}^n$ at k points?

- If k small, points in linearly general position $\rightsquigarrow$ toric varieties, geometry completely described by finite data
- As k grows $\rightsquigarrow$ geometry becomes increasingly intricate
- Rich birational geometry already for $\mathrm{Bl}_k \mathbb{P}^2$
- $n \geq 5$, $k = n + 2, n + 3$ points in linearly general position $\rightsquigarrow$ geometry still described by finite data: $\mathrm{Bl}_k \mathbb{P}^n$ is a **Mori dream space** (non-toric)
- $n \geq 5$, $k = n + 4$, points in **very general position** $\rightsquigarrow \mathrm{Bl}_{n+4} \mathbb{P}^n$ simplest blows-ups which are not Mori dream spaces

SURFACES

$S = \text{Bl}_k \mathbb{P}^2$ blow-up of $\mathbb{P}^2$ at k distinct points

SURFACES

$S = \text{Bl}_k \mathbb{P}^2$ blow-up of $\mathbb{P}^2$ at k distinct points

- If $k \leq 3$, non-collinear points $\rightsquigarrow S$ is toric

SURFACES

$S = \text{Bl}_k \mathbb{P}^2$ blow-up of $\mathbb{P}^2$ at k distinct points

- If $k \leq 3$, non-collinear points $\rightsquigarrow S$ is toric
- If $4 \leq k \leq 8$, general position $\rightsquigarrow S$ is a non-toric del Pezzo surface, geometry still described by finite data, finitely many (-1) -curves

SURFACES

$S = \text{Bl}_k \mathbb{P}^2$ blow-up of $\mathbb{P}^2$ at k distinct points

- If $k \leq 3$, non-collinear points $\rightsquigarrow S$ is toric
- If $4 \leq k \leq 8$, general position $\rightsquigarrow S$ is a non-toric del Pezzo surface, geometry still described by finite data, finitely many (-1) -curves
- If $k \geq 9$, very general position $\rightsquigarrow S$ has infinitely many (-1) -curves

SURFACES

$S = \text{Bl}_k \mathbb{P}^2$ blow-up of $\mathbb{P}^2$ at k distinct points

- If $k \leq 3$, non-collinear points $\rightsquigarrow S$ is toric
- If $4 \leq k \leq 8$, general position $\rightsquigarrow S$ is a non-toric del Pezzo surface, geometry still described by finite data, finitely many (-1) -curves
- If $k \geq 9$, **very general position** $\rightsquigarrow S$ has **infinitely many (-1) -curves**
- So, for points in very general position:

$k \leq 8 \Leftrightarrow S$ del Pezzo $\Leftrightarrow S$ Mori dream $\Leftrightarrow$ finitely many (-1) -curves

SURFACES

$S = \text{Bl}_k \mathbb{P}^2$ blow-up of $\mathbb{P}^2$ at k distinct points

- If $k \leq 3$, non-collinear points $\rightsquigarrow S$ is toric
- If $4 \leq k \leq 8$, general position $\rightsquigarrow S$ is a non-toric del Pezzo surface, geometry still described by finite data, finitely many (-1) -curves
- If $k \geq 9$, very general position $\rightsquigarrow S$ has infinitely many (-1) -curves
- So, for points in very general position:

$$k \leq 8 \Leftrightarrow S \text{ del Pezzo} \Leftrightarrow S \text{ Mori dream} \Leftrightarrow \text{finitely many } (-1)\text{-curves}$$

Very general position = points are in linearly general position after any number of Cremona transformations centered at 3 of the points

SURFACES

$S = \text{Bl}_k \mathbb{P}^2$ blow-up of $\mathbb{P}^2$ at k distinct points

- If $k \leq 3$, non-collinear points $\rightsquigarrow S$ is toric
- If $4 \leq k \leq 8$, general position $\rightsquigarrow S$ is a non-toric del Pezzo surface, geometry still described by finite data, finitely many (-1) -curves
- If $k \geq 9$, **very general position** $\rightsquigarrow S$ has **infinitely many (-1) -curves**
- So, for points in very general position:

$$k \leq 8 \Leftrightarrow S \text{ del Pezzo} \Leftrightarrow S \text{ Mori dream} \Leftrightarrow \text{finitely many } (-1)\text{-curves}$$

Very general position = points are in linearly general position after any number of Cremona transformations centered at 3 of the points

E.g., all points on a conic: after a Cremona centered at 3 of the points, the remaining points lie on a single line $\rightsquigarrow$ finitely many (-1) -curves

FINITE DATA IN HIGHER DIMENSION

X projective variety

FINITE DATA IN HIGHER DIMENSION

X projective variety

To describe the birational geometry of X , we need to understand the **rational contractions** of $X \rightsquigarrow$ controlled by the cones

$$\mathrm{Nef}(X) \subseteq \overline{\mathrm{Mov}}(X) \subseteq \overline{\mathrm{Eff}}(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$$

FINITE DATA IN HIGHER DIMENSION

X projective variety

To describe the birational geometry of X , we need to understand the **rational contractions** of $X \rightsquigarrow$ controlled by the cones

$$\mathrm{Nef}(X) \subseteq \overline{\mathrm{Mov}}(X) \subseteq \overline{\mathrm{Eff}}(X) \subset N^1(X)_{\mathbb{R}}$$

Mori dream spaces (Hu-Keel 2000) are varieties X such that X admits only finitely many rational contractions

FINITE DATA IN HIGHER DIMENSION

X projective variety

To describe the birational geometry of X , we need to understand the **rational contractions** of $X \rightsquigarrow$ controlled by the cones

$$\mathrm{Nef}(X) \subseteq \overline{\mathrm{Mov}}(X) \subseteq \overline{\mathrm{Eff}}(X) \subset N^1(X)_{\mathbb{R}}$$

Mori dream spaces (Hu-Keel 2000) are varieties X such that X admits only finitely many rational contractions

Algebraic characterization:

X is a Mori dream space $\Leftrightarrow$ the **Cox ring** of X is finitely generated

FINITE DATA IN HIGHER DIMENSION

X projective variety

To describe the birational geometry of X , we need to understand the **rational contractions** of $X \rightsquigarrow$ controlled by the cones

$$\mathrm{Nef}(X) \subseteq \overline{\mathrm{Mov}}(X) \subseteq \overline{\mathrm{Eff}}(X) \subset N^1(X)_{\mathbb{R}}$$

Mori dream spaces (Hu-Keel 2000) are varieties X such that X admits only finitely many rational contractions

Algebraic characterization:

X is a Mori dream space $\Leftrightarrow$ the **Cox ring** of X is finitely generated

Examples: all varieties of Fano type (BCHM).

In particular: toric varieties, Fano varieties

MORI CHAMBER DECOMPOSITION

For a Mori dream space X :

(1) The cone of effective divisors $\overline{\text{Eff}}(X)$

- Is generated by finitely many prime divisors
- Decomposes into subcones $\rightsquigarrow$ rational contractions of X

MORI CHAMBER DECOMPOSITION

For a Mori dream space X :

- (1) The cone of effective divisors $\overline{\text{Eff}}(X)$
 - Is generated by finitely many prime divisors
 - Decomposes into subcones $\rightsquigarrow$ rational contractions of X
- (2) The nef cone $\text{Nef}(X)$ is generated by finitely many semiample divisors

MORI CHAMBER DECOMPOSITION

For a Mori dream space X :

- (1) The cone of effective divisors $\overline{\text{Eff}}(X)$
 - Is generated by finitely many prime divisors
 - Decomposes into subcones $\rightsquigarrow$ rational contractions of X
- (2) The nef cone $\text{Nef}(X)$ is generated by finitely many semiample divisors
- (3) The cone of movable divisors $\overline{\text{Mov}}(X)$ decomposes into subcones

$$\overline{\text{Mov}}(X) = \bigcup_i f_i^* \text{Nef}(X_i)$$

corresponding to finitely many small $\mathbb{Q}$ -factorial modifications $f_i : X \dashrightarrow X_i$, such that each X_i has (2).

SET-UP

Points $p_1, \dots, p_k$ very general in $\mathbb{P}^n$; let $X = X_k^n = \text{Bl}_{\{p_i\}} \mathbb{P}^n$

SET-UP

Points $p_1, \dots, p_k$ very general in $\mathbb{P}^n$; let $X = X_k^n = \text{Bl}_{\{p_i\}} \mathbb{P}^n$

- Picard group $\text{Pic}(X) = \mathbb{Z}H \oplus \bigoplus_{i=1}^k \mathbb{Z}E_i$
- Canonical class $K_X = -(n+1)H + (n-1) \sum_i E_i$

SET-UP

Points $p_1, \dots, p_k$ very general in $\mathbb{P}^n$; let $X = X_k^n = \text{Bl}_{\{p_i\}} \mathbb{P}^n$

- Picard group $\text{Pic}(X) = \mathbb{Z}H \oplus \bigoplus_{i=1}^k \mathbb{Z}E_i$
- Canonical class $K_X = -(n+1)H + (n-1) \sum_i E_i$

Very general position $\rightsquigarrow$ yields a Weyl group action on $\text{Pic}(X)$ via Cremona transformations :

SET-UP

Points $p_1, \dots, p_k$ very general in $\mathbb{P}^n$; let $X = X_k^n = \text{Bl}_{\{p_i\}} \mathbb{P}^n$

- Picard group $\text{Pic}(X) = \mathbb{Z}H \oplus \bigoplus_{i=1}^k \mathbb{Z}E_i$
- Canonical class $K_X = -(n+1)H + (n-1) \sum_i E_i$

Very general position $\rightsquigarrow$ yields a Weyl group action on $\text{Pic}(X)$ via Cremona transformations :

$$\mathcal{W}_{n,k} = \langle \phi_I^* \mid I \subset \{1, \dots, k\}, |I| = n+1 \rangle \subset \text{GL}(\text{Pic}(X_k^n))$$

$$\phi_I^* : \text{Pic}(X_k^n) \rightarrow \text{Pic}(X_k^n)$$

involution induced by standard Cremona transformation $\phi_I : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$
centered at the points $\{p_i\}_{i \in I}$

SET-UP

Points $p_1, \dots, p_k$ very general in $\mathbb{P}^n$; let $X = X_k^n = \text{Bl}_{\{p_i\}} \mathbb{P}^n$

- Picard group $\text{Pic}(X) = \mathbb{Z}H \oplus \bigoplus_{i=1}^k \mathbb{Z}E_i$
- Canonical class $K_X = -(n+1)H + (n-1) \sum_i E_i$

Very general position $\rightsquigarrow$ yields a Weyl group action on $\text{Pic}(X)$ via Cremona transformations :

$$\mathcal{W}_{n,k} = \langle \phi_I^* \mid I \subset \{1, \dots, k\}, |I| = n+1 \rangle \subset \text{GL}(\text{Pic}(X_k^n))$$

$$\phi_I^* : \text{Pic}(X_k^n) \rightarrow \text{Pic}(X_k^n)$$

involution induced by standard Cremona transformation $\phi_I : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$
centered at the points $\{p_i\}_{i \in I} \rightsquigarrow$ small modification

$$X \dashrightarrow X' = \text{Bl}_{\{p_j\}_{j \in I} \cup \{\phi_I(p_j)\}_{j \notin I}} \mathbb{P}^n$$

SET-UP

Points $p_1, \dots, p_k$ very general in $\mathbb{P}^n$; let $X = X_k^n = \text{Bl}_{\{p_i\}} \mathbb{P}^n$

- Picard group $\text{Pic}(X) = \mathbb{Z}H \oplus \bigoplus_{i=1}^k \mathbb{Z}E_i$
- Canonical class $K_X = -(n+1)H + (n-1) \sum_i E_i$

Very general position $\rightsquigarrow$ yields a Weyl group action on $\text{Pic}(X)$ via Cremona transformations :

$$\mathcal{W}_{n,k} = \langle \phi_I^* \mid I \subset \{1, \dots, k\}, |I| = n+1 \rangle \subset \text{GL}(\text{Pic}(X_k^n))$$

$$\phi_I^* : \text{Pic}(X_k^n) \rightarrow \text{Pic}(X_k^n)$$

involution induced by standard Cremona transformation $\phi_I : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$
centered at the points $\{p_i\}_{i \in I} \rightsquigarrow$ small modification

$$X \dashrightarrow X' = \text{Bl}_{\{p_j\}_{j \in I} \cup \{\phi_I(p_j)\}_{j \notin I}} \mathbb{P}^n$$

(-1)-divisors: each $E \in \mathcal{W}_{n,k} \cdot E_1$ is an exceptional divisor on some X_k^n

WEYL GROUP AND COBLE PAIRING

Coble Pairing : bilinear form on $\text{Pic}(X)$

$$(H, H) = n-1, \quad (H, E_i) = 0, \quad (E_i, E_j) = -\delta_{ij}$$

WEYL GROUP AND COBLE PAIRING

Coble Pairing : bilinear form on $\text{Pic}(X)$

$$(H, H) = n-1, \quad (H, E_i) = 0, \quad (E_i, E_j) = -\delta_{ij}$$

- For $n = 2$, this is the standard intersection form

WEYL GROUP AND COBLE PAIRING

Coble Pairing : bilinear form on $\text{Pic}(X)$

$$(H, H) = n-1, \quad (H, E_i) = 0, \quad (E_i, E_j) = -\delta_{ij}$$

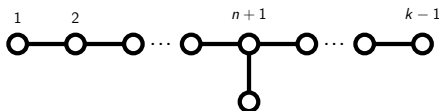
- For $n = 2$, this is the standard intersection form
- There are roots $\alpha_I \in K_X^\perp$ such that $\phi_I = \text{reflection across } \alpha_I^\perp$

WEYL GROUP AND COBLE PAIRING

Coble Pairing : bilinear form on $\text{Pic}(X)$

$$(H, H) = n-1, \quad (H, E_i) = 0, \quad (E_i, E_j) = -\delta_{ij}$$

- For $n = 2$, this is the standard intersection form
- There are roots $\alpha_I \in K_X^\perp$ such that $\phi_I = \text{reflection across } \alpha_I^\perp$
- $\mathcal{W}_{n,k} = \text{Coxeter group of Dynkin-type diagram } T_{2,n+1,k-n-1}$:

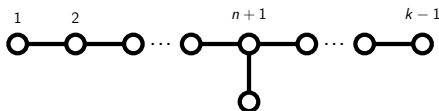


WEYL GROUP AND COBLE PAIRING

Coble Pairing : bilinear form on $\text{Pic}(X)$

$$(H, H) = n-1, \quad (H, E_i) = 0, \quad (E_i, E_j) = -\delta_{ij}$$

- For $n = 2$, this is the standard intersection form
- There are roots $\alpha_I \in K_X^\perp$ such that $\phi_I = \text{reflection across } \alpha_I^\perp$
- $\mathcal{W}_{n,k} = \text{Coxeter group of Dynkin-type diagram } T_{2,n+1,k-n-1}$:



- $\mathcal{W}_{n,k}$ finite $\Leftrightarrow$ ADE type Dynkin diagram

WEYL GROUP AND BIRATIONAL GEOMETRY

$$\mathcal{W}_{n,k} \text{ finite} \iff \begin{cases} n = 2, 4, \ k \leq 8; \text{ or} \\ n = 3, \ k \leq 7; \text{ or} \\ n \geq 5, \ k \leq n + 3. \end{cases}$$

WEYL GROUP AND BIRATIONAL GEOMETRY

$$\mathcal{W}_{n,k} \text{ finite} \iff \begin{cases} n = 2, 4, \ k \leq 8; \text{ or} \\ n = 3, \ k \leq 7; \text{ or} \\ n \geq 5, \ k \leq n + 3. \end{cases}$$

Mukai 2001, 2006, C-Tevelev 2006:

$$X_k^n \text{ Mori dream} \iff \mathcal{W}_{n,k} \text{ finite} \iff \text{finitely many } (-1)\text{-divisors}$$

WEYL GROUP AND BIRATIONAL GEOMETRY

$$\mathcal{W}_{n,k} \text{ finite} \iff \begin{cases} n = 2, 4, & k \leq 8; \text{ or} \\ n = 3, & k \leq 7; \text{ or} \\ n \geq 5, & k \leq n + 3. \end{cases}$$

Mukai 2001, 2006, C-Tevelev 2006:

$$X_k^n \text{ Mori dream} \iff \mathcal{W}_{n,k} \text{ finite} \iff \text{finitely many } (-1)\text{-divisors}$$

Mori chamber decomposition by exploring **Gale duality** :

- Mukai 2006
- Casagrande-Codogni-Fanelli 2019

STRUCTURE INSIDE THE $K_X < 0$ REGION

$X = X_{n+4}^n$ blow-up of $\mathbb{P}^n$ at $n + 4$ points in very general position

Consider the half space $K_X < 0$ in $N^1(X)_{\mathbb{R}}$ w. r. to the Coble pairing

STRUCTURE INSIDE THE $K_X < 0$ REGION

$X = X_{n+4}^n$ blow-up of $\mathbb{P}^n$ at $n + 4$ points in very general position

Consider the half space $K_X < 0$ in $N^1(X)_{\mathbb{R}}$ w. r. to the Coble pairing

MAIN STATEMENT (INFORMAL)

In $K_X < 0$ there is a controlled birational picture with countably many small modifications and blow-downs; the $K_X \leq 0$ part of the moving cone is covered by pullbacks of nef cones of smooth models, which become finitely generated in the strictly $K_X < 0$ part.

(-1) DIVISORS GENERATE EFFECTIVE CONE IN $K_X < 0$

There are countably many small modifications $f_i: X \dashrightarrow X_i$, and birational morphisms $\pi_i: X_i \rightarrow \mathbb{P}^n$ realizing X_i as the blow-up of $\mathbb{P}^n$ at $n+4$ points, such that

$$\overline{\text{Eff}}(X)^{K_X < 0} = \left\langle \bigcup_i f_i^*(\text{Exc}(\pi_i)) \right\rangle,$$

where $\text{Exc}(f_i)$ denotes the set of exceptional divisors of $\pi_i: X_i \rightarrow \mathbb{P}^n$.

MOVING CONE IN $K_X \leq 0$

There are countably many smooth small modifications $g_i: X \dashrightarrow Y_i$ such that

$$\overline{\text{Mov}}(X)^{K_X \leq 0} = \bigcup_i g_i^*(\text{Nef}(Y_i)).$$

MOVING CONE IN $K_X \leq 0$

There are countably many smooth small modifications $g_i: X \dashrightarrow Y_i$ such that

$$\overline{\text{Mov}}(X)^{K_X \leq 0} = \bigcup_i g_i^*(\text{Nef}(Y_i)).$$

If $g_i^*(\text{Nef}(Y_i)) \setminus \{0\} \subset (K_X < 0)$, then the nef cone $\text{Nef}(Y_i)$ is finitely generated.

GALE DUALITY

$k = n + s + 2$, where n, s are non-negative integers

GALE DUALITY

$k = n + s + 2$, where n, s are non-negative integers

There is a bijective correspondence:

$$\left\{ \begin{array}{l} k \text{ points in linearly} \\ \text{general position in } \mathbb{P}^n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} k \text{ points in linearly} \\ \text{general position in } \mathbb{P}^s \end{array} \right\}$$

(all up to projective equivalence).

GALE DUALITY

$k = n + s + 2$, where n, s are non-negative integers

There is a bijective correspondence:

$$\left\{ \begin{array}{l} k \text{ points in linearly} \\ \text{general position in } \mathbb{P}^n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} k \text{ points in linearly} \\ \text{general position in } \mathbb{P}^s \end{array} \right\}$$

(all up to projective equivalence).

$p_1, \dots, p_k$ in $\mathbb{P}^n \rightsquigarrow A = k \times (n + 1)$ rank $(n + 1)$ matrix

$q_1, \dots, q_k$ in $\mathbb{P}^s \rightsquigarrow B = k \times (s + 1)$ rank $(s + 1)$ matrix

GALE DUALITY

$k = n + s + 2$, where n, s are non-negative integers

There is a bijective correspondence:

$$\left\{ \begin{array}{l} k \text{ points in linearly} \\ \text{general position in } \mathbb{P}^n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} k \text{ points in linearly} \\ \text{general position in } \mathbb{P}^s \end{array} \right\}$$

(all up to projective equivalence).

$p_1, \dots, p_k$ in $\mathbb{P}^n \rightsquigarrow A = k \times (n + 1)$ rank $(n + 1)$ matrix

$q_1, \dots, q_k$ in $\mathbb{P}^s \rightsquigarrow B = k \times (s + 1)$ rank $(s + 1)$ matrix

The configurations $\{p_i\}$ and $\{q_i\}$ are Gale dual if

$$B^t \cdot D \cdot A = 0$$

for some diagonal matrix D (so for some choice of coordinates $B^t \cdot A = 0$)

GALE DUALITY - SPECIAL CASES

Special case $s = 1$:

$$\{ q_1, \dots, q_{n+3} \text{ in } \mathbb{P}^1 \} \longleftrightarrow \{ p_1, \dots, p_{n+3} \text{ in } \mathbb{P}^n \}$$

$\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ rational normal curve of degree n , $p_i = \phi(q_i)$

GALE DUALITY - SPECIAL CASES

Special case $s = 1$:

$$\{ q_1, \dots, q_{n+3} \text{ in } \mathbb{P}^1 \} \longleftrightarrow \{ p_1, \dots, p_{n+3} \text{ in } \mathbb{P}^n \}$$

$\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ rational normal curve of degree n , $p_i = \phi(q_i)$

Special case $s = 2$:

$$\{ q_1, \dots, q_{n+4} \text{ in } \mathbb{P}^2 \} \longleftrightarrow \{ p_1, \dots, p_{n+4} \text{ in } \mathbb{P}^n \}$$

GALE DUAL BLOW-UPS

$$X_k^s = \text{Bl}_{q_1, \dots, q_k} \mathbb{P}^s, \quad X_k^n = \text{Bl}_{p_1, \dots, p_k} \mathbb{P}^n \quad s, n \geq 2$$

GALE DUAL BLOW-UPS

$$X_k^s = \text{Bl}_{q_1, \dots, q_k} \mathbb{P}^s, \quad X_k^n = \text{Bl}_{p_1, \dots, p_k} \mathbb{P}^n \quad s, n \geq 2$$

Canonical isometry $K_{X_k^s}^\perp \rightarrow K_{X_k^n}^\perp$ with respect to the Coble pairing

$\rightsquigarrow$ can identify the Weyl groups $\mathcal{W}_{s,k} \cong \mathcal{W}_{n,k}$

GALE DUAL BLOW-UPS

$$X_k^s = \text{Bl}_{q_1, \dots, q_k} \mathbb{P}^s, \quad X_k^n = \text{Bl}_{p_1, \dots, p_k} \mathbb{P}^n \quad s, n \geq 2$$

Canonical isometry $K_{X_k^s}^\perp \rightarrow K_{X_k^n}^\perp$ with respect to the Coble pairing

$\rightsquigarrow$ can identify the Weyl groups $\mathcal{W}_{s,k} \cong \mathcal{W}_{n,k}$

QUESTION

How are the geometries of X_k^s and X_k^n related?

GALE DUAL BLOW-UPS

$$X_k^s = \text{Bl}_{q_1, \dots, q_k} \mathbb{P}^s, \quad X_k^n = \text{Bl}_{p_1, \dots, p_k} \mathbb{P}^n \quad s, n \geq 2$$

Canonical isometry $K_{X_k^s}^\perp \rightarrow K_{X_k^n}^\perp$ with respect to the Coble pairing

$\rightsquigarrow$ can identify the Weyl groups $\mathcal{W}_{s,k} \cong \mathcal{W}_{n,k}$

QUESTION

How are the geometries of X_k^s and X_k^n related?

Works even for $s = 1$ (Mukai 2006):

GALE DUAL BLOW-UPS

$$X_k^s = \text{Bl}_{q_1, \dots, q_k} \mathbb{P}^s, \quad X_k^n = \text{Bl}_{p_1, \dots, p_k} \mathbb{P}^n \quad s, n \geq 2$$

Canonical isometry $K_{X_k^s}^\perp \rightarrow K_{X_k^n}^\perp$ with respect to the Coble pairing

$\rightsquigarrow$ can identify the Weyl groups $\mathcal{W}_{s,k} \cong \mathcal{W}_{n,k}$

QUESTION

How are the geometries of X_k^s and X_k^n related?

Works even for $s = 1$ (Mukai 2006):

$X_{n+3}^n = \text{Bl}_{n+3} \mathbb{P}^n$ is a **moduli space of semistable rank 2 parabolic bundles on $\mathbb{P}^1$** with parabolic structure at the (Gale dual) $n+3$ points (Bauer 1991)

GALE DUAL BLOW-UPS

$$X_k^s = \text{Bl}_{q_1, \dots, q_k} \mathbb{P}^s, \quad X_k^n = \text{Bl}_{p_1, \dots, p_k} \mathbb{P}^n \quad s, n \geq 2$$

Canonical isometry $K_{X_k^s}^\perp \rightarrow K_{X_k^n}^\perp$ with respect to the Coble pairing

$\rightsquigarrow$ can identify the Weyl groups $\mathcal{W}_{s,k} \cong \mathcal{W}_{n,k}$

QUESTION

How are the geometries of X_k^s and X_k^n related?

Works even for $s = 1$ (Mukai 2006):

$X_{n+3}^n = \text{Bl}_{n+3} \mathbb{P}^n$ is a **moduli space of semistable rank 2 parabolic bundles on $\mathbb{P}^1$** with parabolic structure at the (Gale dual) $n+3$ points (Bauer 1991)

$\rightsquigarrow$ Mori chamber decomposition using variation of stability (weights)

SPECIAL CASE $k = n + 4$ OF GALE DUALITY

$$S = X_{n+4}^2 = \mathrm{Bl}_{q_1, \dots, q_{n+4}} \mathbb{P}^2, \quad X = X_{n+4}^n = \mathrm{Bl}_{p_1, \dots, p_{n+4}} \mathbb{P}^n$$

SPECIAL CASE $k = n + 4$ OF GALE DUALITY

$$S = X_{n+4}^2 = \mathrm{Bl}_{q_1, \dots, q_{n+4}} \mathbb{P}^2, \quad X = X_{n+4}^n = \mathrm{Bl}_{p_1, \dots, p_{n+4}} \mathbb{P}^n$$

Mukai 2006, Casagrande-Codogni-Fanelli 2019 studied the case when S is a del Pezzo surface ($n = 2, 3, 4$)

SPECIAL CASE $k = n + 4$ OF GALE DUALITY

$$S = X_{n+4}^2 = \mathrm{Bl}_{q_1, \dots, q_{n+4}} \mathbb{P}^2, \quad X = X_{n+4}^n = \mathrm{Bl}_{p_1, \dots, p_{n+4}} \mathbb{P}^n$$

Mukai 2006, Casagrande-Codogni-Fanelli 2019 studied the case when S is a del Pezzo surface ($n = 2, 3, 4$)

For all $n \geq 2$: choose an ample polarization L on S . We consider

$$M_L = \{\text{rank-2 torsion-free } F \text{ on } S : c_1(F) = -K_S, \ c_2(F) = 2\}$$

moduli space of Gieseker L -semistable sheaves on S

SPECIAL CASE $k = n + 4$ OF GALE DUALITY

$$S = X_{n+4}^2 = \mathrm{Bl}_{q_1, \dots, q_{n+4}} \mathbb{P}^2, \quad X = X_{n+4}^n = \mathrm{Bl}_{p_1, \dots, p_{n+4}} \mathbb{P}^n$$

Mukai 2006, Casagrande-Codogni-Fanelli 2019 studied the case when S is a del Pezzo surface ($n = 2, 3, 4$)

For all $n \geq 2$: choose an ample polarization L on S . We consider

$$M_L = \{\text{rank-2 torsion-free } F \text{ on } S : c_1(F) = -K_S, \ c_2(F) = 2\}$$

moduli space of Gieseker L -semistable sheaves on S

THEOREM

There exists an ample line bundle L on S , explicitly determined, such that

$$X \cong M_L.$$

GALE DUALITY AND MODULI

$$S = X_{n+4}^2 = \mathrm{Bl}_{q_1, \dots, q_{n+4}} \mathbb{P}^2, \quad X = X_{n+4}^n = \mathrm{Bl}_{p_1, \dots, p_{n+4}} \mathbb{P}^n$$

THEOREM

There exists an ample line bundle L on S , explicitly determined, such that

$$X \cong M_L = \{\text{rank-2 torsion-free } F \text{ on } S : c_1(F) = -K_S, \ c_2(F) = 2\}$$

moduli space of Gieseker L -semistable sheaves on S .

GALE DUALITY AND MODULI

$$S = X_{n+4}^2 = \mathrm{Bl}_{q_1, \dots, q_{n+4}} \mathbb{P}^2, \quad X = X_{n+4}^n = \mathrm{Bl}_{p_1, \dots, p_{n+4}} \mathbb{P}^n$$

THEOREM

There exists an ample line bundle L on S , explicitly determined, such that

$$X \cong M_L = \{\text{rank-2 torsion-free } F \text{ on } S : c_1(F) = -K_S, \ c_2(F) = 2\}$$

moduli space of Gieseker L -semistable sheaves on S .

Idea: There is a $\mathbb{P}^n$ parametrizing non-trivial extensions of the form

$$0 \rightarrow \mathcal{O}_S(-K - H) \rightarrow F \rightarrow \mathcal{O}_S(H) \rightarrow 0.$$

GALE DUALITY AND MODULI

$$S = X_{n+4}^2 = \mathrm{Bl}_{q_1, \dots, q_{n+4}} \mathbb{P}^2, \quad X = X_{n+4}^n = \mathrm{Bl}_{p_1, \dots, p_{n+4}} \mathbb{P}^n$$

THEOREM

There exists an ample line bundle L on S , explicitly determined, such that

$$X \cong M_L = \{\text{rank-2 torsion-free } F \text{ on } S : c_1(F) = -K_S, \ c_2(F) = 2\}$$

moduli space of Gieseker L -semistable sheaves on S .

Idea: There is a $\mathbb{P}^n$ parametrizing non-trivial extensions of the form

$$0 \rightarrow \mathcal{O}_S(-K - H) \rightarrow F \rightarrow \mathcal{O}_S(H) \rightarrow 0.$$

The variation of the moduli spaces M_L when we vary the polarization L on S is well understood, particularly when $L \cdot K < 0$:

Friedman-Qin ('95), Göttsche ('96), Matsuki-Wentworth ('97)

WHAT STABILITY BUYS YOU

- For general L with $K_S \cdot L < 0$, the moduli spaces M_L are birational models of $X = X_{n+4}^n$ which are smooth, irreducible, rational varieties of dimension n parametrizing **vector bundles** on S

WHAT STABILITY BUYS YOU

- For general L with $K_S \cdot L < 0$, the moduli spaces M_L are birational models of $X = X_{n+4}^n$ which are smooth, irreducible, rational varieties of dimension n parametrizing **vector bundles** on S
- **Determinant line bundle** $\mathcal{L}_L$ on M_L controls ample classes and maps

WHAT STABILITY BUYS YOU

- For general L with $K_S \cdot L < 0$, the moduli spaces M_L are birational models of $X = X_{n+4}^n$ which are smooth, irreducible, rational varieties of dimension n parametrizing **vector bundles** on S
- **Determinant line bundle** $\mathcal{L}_L$ on M_L controls ample classes and maps
- **Stability chambers** in $\{K_S \leq 0\} \rightsquigarrow$ **Mori chambers** in $\{K_X \leq 0\}$

WHAT STABILITY BUYS YOU

- For general L with $K_S \cdot L < 0$, the moduli spaces M_L are birational models of $X = X_{n+4}^n$ which are smooth, irreducible, rational varieties of dimension n parametrizing **vector bundles** on S
- **Determinant line bundle** $\mathcal{L}_L$ on M_L controls ample classes and maps
- **Stability chambers in $\{K_S \leq 0\}$** $\rightsquigarrow$ **Mori chambers in $\{K_X \leq 0\}$**
- **(The determinant map)** There is an isomorphism

$$\rho : N^1(S)_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}}$$

such that

$$\rho(K_S) = K_X, \quad \rho(K_S \leq 0) = (K_X \leq 0),$$

WHAT STABILITY BUYS YOU

- For general L with $K_S \cdot L < 0$, the moduli spaces M_L are birational models of $X = X_{n+4}^n$ which are smooth, irreducible, rational varieties of dimension n parametrizing **vector bundles** on S
- **Determinant line bundle** $\mathcal{L}_L$ on M_L controls ample classes and maps
- **Stability chambers in $\{K_S \leq 0\}$** $\rightsquigarrow$ **Mori chambers in $\{K_X \leq 0\}$**
- **(The determinant map)** There is an isomorphism

$$\rho : N^1(S)_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}}$$

such that

$$\rho(K_S) = K_X, \quad \rho(K_S \leq 0) = (K_X \leq 0),$$

$$(1/2)\rho \text{ is the canonical isometry } K_S^{\perp} \rightarrow K_X^{\perp}$$

WHAT STABILITY BUYS YOU

- For general L with $K_S \cdot L < 0$, the moduli spaces M_L are birational models of $X = X_{n+4}^n$ which are smooth, irreducible, rational varieties of dimension n parametrizing **vector bundles** on S
- **Determinant line bundle** $\mathcal{L}_L$ on M_L controls ample classes and maps
- **Stability chambers in $\{K_S \leq 0\}$** $\rightsquigarrow$ **Mori chambers in $\{K_X \leq 0\}$**
- **(The determinant map)** There is an isomorphism

$$\rho : N^1(S)_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}}$$

such that

$$\rho(K_S) = K_X, \quad \rho(K_S \leq 0) = (K_X \leq 0),$$

$$(1/2)\rho \text{ is the canonical isometry } K_S^{\perp} \rightarrow K_X^{\perp}$$

For L ample, general: $\rho(L) = \mathcal{L}_L =$ determinant line bundle on M_L

NUMERICALLY RATIONAL DIVISOR CLASSES

A divisor D on S is **numerically rational** if

$$D^2 + D \cdot K_S = -2$$

NUMERICALLY RATIONAL DIVISOR CLASSES

A divisor D on S is **numerically rational** if

$$D^2 + D \cdot K_S = -2$$

Lines/conics/cubics correspond to $D^2 = -1, 0, 1$ via $D \cdot (-K_S) = 1, 2, 3$

NUMERICALLY RATIONAL DIVISOR CLASSES

A divisor D on S is **numerically rational** if

$$D^2 + D \cdot K_S = -2$$

Lines/conics/cubics correspond to $D^2 = -1, 0, 1$ via $D \cdot (-K_S) = 1, 2, 3$

Numerical rational classes D with

$$-n \leq D^2 \leq 1$$

control stability in $\{K_S \leq 0\}$:

The only walls in $\{K_S \leq 0\}$ where stability changes are $(2D + K_S)^\perp$

NUMERICALLY RATIONAL DIVISOR CLASSES

A divisor D on S is **numerically rational** if

$$D^2 + D \cdot K_S = -2$$

Lines/conics/cubics correspond to $D^2 = -1, 0, 1$ via $D \cdot (-K_S) = 1, 2, 3$

Numerical rational classes D with

$$-n \leq D^2 \leq 1$$

control stability in $\{K_S \leq 0\}$:

The only walls in $\{K_S \leq 0\}$ where stability changes are $(2D+K_S)^\perp$

Up to the action of the Weyl group, lines/conics/cubics cutting walls in $\{K_S \leq 0\}$ are given by

$$D = E_i, \quad D = H - E_i \quad D = H$$

LOCAL MODEL OF A WALL CROSSING

Neighbouring chambers C_- and C_+ across $(2D+K_S)^\perp$:

LOCAL MODEL OF A WALL CROSSING

Neighbouring chambers C_- and C_+ across $(2D+K_S)^\perp$:

- Special loci $\mathbb{P}_D \cong \mathbb{P}^{-D^2}$ and $\mathbb{P}_{-K_S-D} \cong \mathbb{P}^{n+D^2-1} \rightsquigarrow$ extensions

$$0 \rightarrow \mathcal{O}_S(D) \rightarrow E \rightarrow \mathcal{O}_S(-K-D) \rightarrow 0$$

$$0 \rightarrow \mathcal{O}_S(-K-D) \rightarrow E \rightarrow \mathcal{O}_S(D) \rightarrow 0$$

.

LOCAL MODEL OF A WALL CROSSING

Neighbouring chambers C_- and C_+ across $(2D+K_S)^\perp$:

- Special loci $\mathbb{P}_D \cong \mathbb{P}^{-D^2}$ and $\mathbb{P}_{-K_S-D} \cong \mathbb{P}^{n+D^2-1} \rightsquigarrow$ extensions

$$0 \rightarrow \mathcal{O}_S(D) \rightarrow E \rightarrow \mathcal{O}_S(-K-D) \rightarrow 0$$

$$0 \rightarrow \mathcal{O}_S(-K-D) \rightarrow E \rightarrow \mathcal{O}_S(D) \rightarrow 0$$

.

- Birational transform/explicit MMP between models:

$$\mathrm{Bl}_{\mathbb{P}_D} \mathcal{M}_{C_-} \cong \mathrm{Bl}_{\mathbb{P}_{-K_S-D}} \mathcal{M}_{C_+}.$$

LOCAL MODEL OF A WALL CROSSING

Neighbouring chambers C_- and C_+ across $(2D+K_S)^\perp$:

- Special loci $\mathbb{P}_D \cong \mathbb{P}^{-D^2}$ and $\mathbb{P}_{-K_S-D} \cong \mathbb{P}^{n+D^2-1} \rightsquigarrow$ extensions

$$0 \rightarrow \mathcal{O}_S(D) \rightarrow E \rightarrow \mathcal{O}_S(-K-D) \rightarrow 0$$

$$0 \rightarrow \mathcal{O}_S(-K-D) \rightarrow E \rightarrow \mathcal{O}_S(D) \rightarrow 0$$

.

- Birational transform/explicit MMP between models:

$$\mathrm{Bl}_{\mathbb{P}_D} \mathcal{M}_{C_-} \cong \mathrm{Bl}_{\mathbb{P}_{-K_S-D}} \mathcal{M}_{C_+}.$$

- For $-(n-2) \leq D^2 \leq -1$, flip/flop/anti-flip according to D^2 vs. $\frac{1-n}{2}$

LOCAL MODEL OF A WALL CROSSING

Neighbouring chambers C_- and C_+ across $(2D+K_S)^\perp$:

- Special loci $\mathbb{P}_D \cong \mathbb{P}^{-D^2}$ and $\mathbb{P}_{-K_S-D} \cong \mathbb{P}^{n+D^2-1} \rightsquigarrow$ extensions

$$0 \rightarrow \mathcal{O}_S(D) \rightarrow E \rightarrow \mathcal{O}_S(-K-D) \rightarrow 0$$

$$0 \rightarrow \mathcal{O}_S(-K-D) \rightarrow E \rightarrow \mathcal{O}_S(D) \rightarrow 0$$

.

- Birational transform/explicit MMP between models:

$$\mathrm{Bl}_{\mathbb{P}_D} \mathcal{M}_{C_-} \cong \mathrm{Bl}_{\mathbb{P}_{-K_S-D}} \mathcal{M}_{C_+}.$$

- For $-(n-2) \leq D^2 \leq -1$, flip/flop/anti-flip according to D^2 vs. $\frac{1-n}{2}$
- For $D^2 = -(n-1), 0$, blow-downs of the form $X_{n+4}^n \rightarrow \mathbb{P}^n$

Consider the following closed convex cones:

$$\Pi = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2C + K) \geq 0 \text{ for all numerical conics } C\}$$

Consider the following closed convex cones:

$$\Pi = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2C + K) \geq 0 \text{ for all numerical conics } C\}$$

$$\mathcal{E} = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2B + K) \geq 0 \text{ for all numerical cubics } B\}$$

Consider the following closed convex cones:

$$\Pi = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2C + K) \geq 0 \text{ for all numerical conics } C\}$$

$$\mathcal{E} = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2B + K) \geq 0 \text{ for all numerical cubics } B\}$$

The Weyl group $\mathcal{W}$ acts on Π and $\mathcal{E}$

Up to the action of the Weyl group, a numerical cubic has the form $B = H = C + E_i$, for the numerical conic $C = H - E_i$, therefore:

$$\Pi \subset \mathcal{E} \subset \text{Nef}(S)^{K \leq 0}$$

Consider the following closed convex cones:

$$\Pi = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2C + K) \geq 0 \text{ for all numerical conics } C\}$$

$$\mathcal{E} = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2B + K) \geq 0 \text{ for all numerical cubics } B\}$$

The Weyl group $\mathcal{W}$ acts on Π and $\mathcal{E}$

Up to the action of the Weyl group, a numerical cubic has the form $B = H - E_i$, for the numerical conic $C = H - E_i$, therefore:

$$\Pi \subset \mathcal{E} \subset \text{Nef}(S)^{K \leq 0}$$

Stability chambers in $\Pi \rightsquigarrow$ Mori chambers in $\overline{\text{Mov}}(X)$

Consider the following closed convex cones:

$$\Pi = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2C + K) \geq 0 \text{ for all numerical conics } C\}$$

$$\mathcal{E} = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2B + K) \geq 0 \text{ for all numerical cubics } B\}$$

The Weyl group $\mathcal{W}$ acts on Π and $\mathcal{E}$

Up to the action of the Weyl group, a numerical cubic has the form $B = H - E_i$, for the numerical conic $C = H - E_i$, therefore:

$$\Pi \subset \mathcal{E} \subset \text{Nef}(S)^{K \leq 0}$$

Stability chambers in $\Pi \rightsquigarrow$ Mori chambers in $\overline{\text{Mov}}(X)$

Stability chambers in $\mathcal{E} \rightsquigarrow$ Mori chambers in $\overline{\text{Eff}}(X)$

$$\Pi = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2C + K) \geq 0 \text{ for all numerical conics } C\}$$

$$\mathcal{E} = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2B + K) \geq 0 \text{ for all numerical cubics } B\}$$

$$\Pi \subset \mathcal{E} \subset \text{Nef}(S)^{K \leq 0}$$

$$\Pi = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2C + K) \geq 0 \text{ for all numerical conics } C\}$$

$$\mathcal{E} = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2B + K) \geq 0 \text{ for all numerical cubics } B\}$$

$$\Pi \subset \mathcal{E} \subset \text{Nef}(S)^{K \leq 0}$$

THEOREM

Consider the determinant map

$$\rho : N^1(S)_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}}$$

$$\Pi = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2C + K) \geq 0 \text{ for all numerical conics } C\}$$

$$\mathcal{E} = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2B + K) \geq 0 \text{ for all numerical cubics } B\}$$

$$\Pi \subset \mathcal{E} \subset \text{Nef}(S)^{K \leq 0}$$

THEOREM

Consider the determinant map

$$\rho : N^1(S)_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}}$$

Then

$$\rho(\Pi) = \overline{\text{Mov}}(X)^{K_X \leq 0}, \quad \rho(\mathcal{E}) = \overline{\text{Eff}}(X)^{K_X \leq 0}.$$

$$\Pi = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2C + K) \geq 0 \text{ for all numerical conics } C\}$$

$$\mathcal{E} = \{L \in \text{Nef}(S) \mid L \cdot K \leq 0, L \cdot (2B + K) \geq 0 \text{ for all numerical cubics } B\}$$

$$\Pi \subset \mathcal{E} \subset \text{Nef}(S)^{K \leq 0}$$

THEOREM

Consider the determinant map

$$\rho : N^1(S)_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}}$$

Then

$$\rho(\Pi) = \overline{\text{Mov}}(X)^{K_X \leq 0}, \quad \rho(\mathcal{E}) = \overline{\text{Eff}}(X)^{K_X \leq 0}.$$

For any chamber $C \subset \Pi$, we have

$$\rho(\overline{C}) = \text{Nef}(M_C) \subset N^1(M_C)_{\mathbb{R}} = N^1(X)_{\mathbb{R}}$$